Topology

Course Outline

1. Review of set theory
2. Metric space
3. Neighbourhood, closed sets, open sets.
4. Topological Spaces
5. Bases and subspaces.
6. Continuity in top spaces
7. Separation axioms
8. Homomorphism
9. Relative topologies
10. Countable and uncountable sets.

SET THEORY

Set - well defined collection of distinct objects.

Members of a set are denoted by small letters while sets are devoted by capital letters e.g Sets are labelled A, B, C, D.. Objects (members) are a, b, c, d

Sets are ~~rep~~ Rep by Cal. brackets and members listed in terms of weight and separated using commas e.g.

$A = \{1, 2, 3, 4, 5\}$

Notations

$\in$ - belong to

$\exists$ - There exist

$\ni$ - Such that.

$\Rightarrow$ Implies

$\Leftarrow$ implied by

$\Leftrightarrow$ If and only if

$||$ - modulus | Determinant | Cardinality

Types of sets

1. Empty set - set without element $\emptyset$ or $\{\}$

- It's unique
- subset of any other set.

Examples of null set

i) $A \Rightarrow \{x : x \neq x\}$

Singleton set - A set containing one element only.

$B = \{m : m \text{ is a flying mammal}\}$

$X = \{\emptyset\}$

A set of real no.

$\mathbb{N}$ - natural

$\mathbb{Q}$ - Rational

$\mathbb{Z}$ - intergers

$\mathbb{R}$ - Real no.

$\mathbb{C}$ - complex.

A set containing other sets is called a family or class of sets.

Classes are denoted using script letters.

Script letters

A - $\mathcal{A}$
B - $\mathcal{B}$
C - $\mathcal{C}$
D = $\mathcal{D}$
E - $\mathcal{E}$
F - $\mathcal{F}$
G - $\mathcal{G}$
H - $\mathcal{H}$
I - $\mathcal{I}$
J - $\mathcal{J}$
K - $\mathcal{K}$
L - $\mathcal{L}$
M - $\mathcal{M}$
N - $\mathcal{N}$
O - $\mathcal{O}$
P - $\mathcal{P}$
Q - $\mathcal{Q}$
R - $\mathcal{R}$
S - $\mathcal{S}$
T - $\mathcal{T}$
U - $\mathcal{U}$
V - $\mathcal{V}$
W - $\mathcal{W}$
X - $\mathcal{X}$
Y - $\mathcal{Y}$
Z - $\mathcal{Z}$

Universal sets (U)

Is called universal set if it contains elements in universal property however elements can be divide into subset with specific non universal properties

$A = \{a, b, c - - - z\}$

$V = \{a, e, i, o, u\}$

$C = \{c : c$ is a constant$\}$

$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$O = \{1, 3, 5, 7, 9\}$

$E = \{0, 2, 4, 6, 8\}$

$P = \{2, 3, 7\}$

A set is said to be finite if you can exhaust all its members by counting. If a set is non finite then we say its infinite.

Power Set

let A be a finite set. The power set of A denoted by $P(A)$ is the set containing all the subset of A.

A power set contains 2^n elements where n is no of elements in A.

If A has 3 elements then the power set of A will contain $2^3 = 8$.

Subset

A set A is called a subset of B if all elements of A are in B. We denote it by

$A \subseteq B$

$\exists$ atleast one element in B which is not in A then we say A is a proper subset of B denoted by

$A \subset B$

Equal Sets

A set $A = B$ if A is a subset of B ~~and~~ B is a subset of A.

Exercise

Define comparable sets and give examples.

If $A = \{1, 2, 3\}$ $B = \{0, 2, 4\}$

$U = \{0, 1, 2, 3, 4, 5\}$.

if $A \nsubseteq B$ $B \nsubseteq A$.

$\Rightarrow$ A set is a subset of itself
An empty set is a set of any other set.

Subset of A are.

$\{\emptyset, A, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

$2^3 = 8$

If $A = \{\}$

Then $P(P(P(A)))$ has has how many members.

$P[A]$ has 2^n elements.

$2^0 = 1$

$P(1)$ has 2^1 element $= 2$

$P(2)$ has 2^2 elements.

$2^2 = 4$

Set Complimentation

1. Universal Complimentation.

let U be a universal set and A a Subset of U then Compliment of A is denoted by A^c

$U / A = \{x : x \in U \; x \notin A\}$

$A^c = U/A = \{0,1,2,3,4,5\}/\{0,2,3\}$

$= \{1, 4, 5\}$

NOTE $(A^c)^c = A$.

$U^c = \phi$

$\phi^c = U$

ii) Relative Compliment.

Given the sets A and B

$A/B = \{x : x \in A, \; x \notin B\}$

e.g if $A = \{1,2,3\}$, $B = \{0,1,4\}$

$A/B = \{2,3\}$

$B/A = \{0,4\}$

Other set Operations.

Union

Given 2 sets A and B the U $A \cup B$ is the set $= \{x : x \in A$ or $x \in B\}$.

Example

$A = \{0,1,2,3\}$ $B = \{1,4,5\}$

$A \cup B = \{0,1,2,3,4,5\}$.

Intersection

Given 2 set A and B

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A \cap B = \{1\}$

Symmetric difference.

$A \Delta B = A/B \cup B/A$.

e.g if $A = \{0,2,3,4\}$, $B = \{0,1,3,5\}$.

find $A \Delta B$.

$A/B = \{2,4\}$

$B/A = \{1,5\}$

$A/B \cup B/A$.

$\{1,2,4,5\}$

Assignment

State and proof all the Algebraic Laws of Sets.

State and prove the De morgan theorem

Define giveing examples of the Cartesian product of sets.

let ~~$x \in A^c$~~ an $x \in A^c \cap B^c$

$\Rightarrow x \in A^c$ and $x \in B^c$

$x \notin A$ or $x \notin B$.

$x \notin A \cup B$

$\Rightarrow x \in (A \cup B)^c$

Cardinality of sets

Cardinality of A refers to no of elements in A.

Denoted by Card (A)

$n(A)$

$|A|$

For instance if $A = \{1,2,3,4\}$.

Card (A) = 4

If $A = \{ y : y \neq y \}$

Card (A) = 0

$B = \{\{\}\}$

$= 1$

$C = \{\{\}, \{\}, \phi\}$

$= 1$

Assignment

Let $U = \{0, 1, 2 - - - 9\}$

$A = \{0, 1, 4, 5\}$

$B = \{1, 3, 5\}$

$C = \{3, 4, 6, 7, 9\}$

Find

i) A^c, B^c, C^c, U^c

ii) $A \cup B$, $A \cup C$, $B \cup C$

iii) $A \cap B$, $A \cap C$, $B \cap C$

iv) Compliments of sets in (ii) and (iii)

v) power sets of sets in (ii), iii, iv,

vi) Cardinality of sets in ii, iii, iv, v

vii) Symmetric difference of all possible unique pairs in ii – vi

viii) $\{([A^c \cup B^c]^c \cap (A \cap B \cap C^c)^c)^c\}$

Exercise

State and prove all the Algebraic laws of Sets

1. Idempotent Laws

* let A, B, C be arbitrary subsets of a universal set.

Proof

$A \cup A = A$

$A \cap A = A$ $\quad x \in A \cup A \Rightarrow x \in A$.

2. Identity Laws.

$A \cup \phi = A$ = union of sets adds nothing

$A \cap U = A$ = intersection with universal sets keeps everything in A.

3. Domination Laws.

$A \cup U = U$ = Any element in U is in $A \cup U$

$A \cap \phi = \phi$ - No element exists in both A and ϕ

4. Compliment Laws.

$A \cup A' = U$ every element is either A or not.

$A \cap A' = \phi$ No element can be both in A and A'

5. Commutative Laws.

$A \cup B = B \cup A$

$A \cap B = B \cap A$.

6. Associative Laws.

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

7. Distributive Laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

8. Absorption laws

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$.

9. Demorgans Law.

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

10. Double Compliment law

$(A')' = A$.

Cardinality theorems

i Two Sets

Given two sets A and B.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example

In WMB 9404 Class 80% use black biro 90% - blue, 75% both. Determine the % using either blue or black.

$|B_1| = 80$

$|B_2| = 90$

$|B_1 \cap B_2| = 75$

$|B_1| + |B_2| = (80 + 90)$

$(80 + 90) - 75 = 95$

ii) 3 sets

Generalised cardinality theorem
A, B, C

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cup B| + |A \cup C| + |B \cup C|)$$

*Describe the application of topology in real life.

1) Computer Science & Data analysis
- Topological data analysis.

2. Robotic & motion Planning
- Helps robots plan paths avoid Obstacles.

3. Biology
- DNA structure.
- Neuroscience.

4. Physics
topological phases of matter.

5. Geography & GPS

6. Economics.

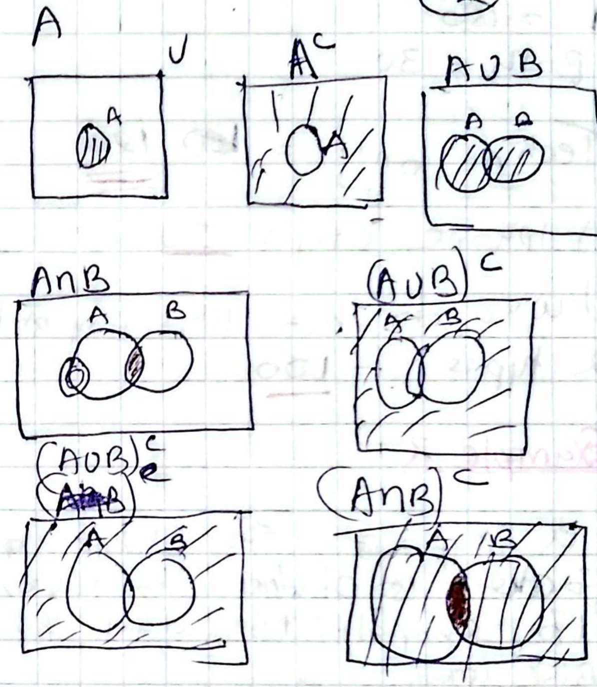
VENN DIAGRAMS

A pictorial rep of sets using rectangles and circles the circles must be inside the rectangles.

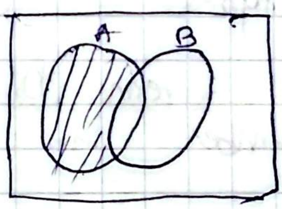
Circles - subset or universal
Rectangle - universal

NOTE

We always shed the required parts.



$A \cap B^c$



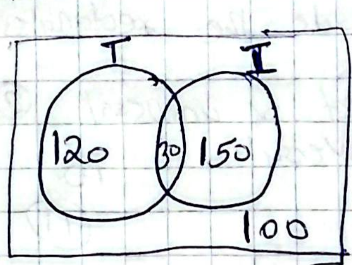
Example 1

In topology Class of 400 Students ~~300~~ 150 Students like techno, 180 like iphone while 30 like both techno and iphone.

i) Rep the info. above in a venn diagram.

ii) Determine the no of Students that like techno only - ~~12~~

Solution



$U = 400$
$T = 150$
$I = 150$
Both $= 30$.

Techno only $=$ ~~150~~ $\underline{120}$

iii) Iphone only $= \underline{150}$.

iv) who do not like any of the 2 types $- \underline{100}$

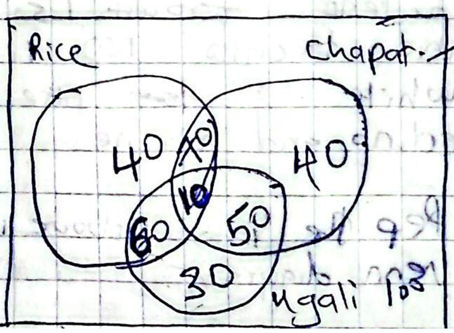
Example 2

In a group of 500 Students at Jooust 180 students like rice, 170 - Chapati; 150, ugali, 70. Rice ugali, 80 - Both rice and Chapati 60 - both chapat and ugali 10 - All 3 types.

- present on a venn Diagrams
- find the number that likes

i) Rice only.
ii) Chapati only.
iii) ugali only $= 10$
iv) Does not like and of 3.

170



Matric spaces

Cartesian Products of Sets:

Given 2 non-empty sets, the Cartesian product of A and B denoted $A \times B = \{(a, y) \; a \in A$ and $y \in B\}$

You can have a Cartesian product of a Set Itself.

Example

Given that $A = \{1\ 2\ 3\ 4\}$
$B = \{a, b, c\}$

find $A \times B$

$= [(1,a)(1,b)(1,c)(2,a)(2,b)$
$(2,c)(3,a)(3,b)(3,c)(4,a)$
$(4,b),(4,c)]$

Metric Space

Let X be a metric space. A non-negative real value function. $d: X \times X \to \mathbb{R}$ is called a metric of $\mathbb{R}$ if the following conditions are satisfied.

1/ Non-negativity.

$d(x,y) \geq 0, \forall x,y \in X$

2/ Zero property

$d(x,y) = 0$ if $x = y$

3/ Symmetry $= d(x,y) = d(y,x)$

IV/ Triangle property $d(x,z) \leq d(x,y) + d(y,z)$

$\forall x,y,z \in X$

The orderered pair (X,d) is what we call a metric space.

NOTE

If property 2 is not satisfied then (X,d) is a called a ~~trigonometric~~ Psendometric Space.

Example

Let $X = \mathbb{R}$

$d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$d(x,y) = |x-y|, \forall x,y \in \mathbb{R}$

Show that d is a metric on $\mathbb{R}$.

Soln

1 Clearly

1) $d(x,y) \geq 0, \forall x,y \in \mathbb{R}$

$d(x,y) = |x-y| \geq 0$

ii) $|x-y| = x-y = 0$ iff $y = x$

So $d(x,y) = 0$ iff $x = y$

iii) $d(x,y) = |x-y| = |-(y-x)|$

$= |y-x|$

$= d(y,x)$

So $d(x,y) = d(y,x)$

IV) $d(x,z) \leq d(x,y) + d(y,z)$

$d(x,z) = |x-z|$

$= |x-y+y-z|$

$\leq |x-y| + |y-z|$

$d(x,y) + d(y,z)$

$d(x,z) \leq d(x,y) + d(y,z)$

Hence $d(x,y) = |x-y|$ is a metric on $\mathbb{R}$.

Example 4

let $X = n$

as $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $d(x,y) = |x^2 - y^2|$ $\forall x,y \in \mathbb{R}$

Is d metric on $\mathbb{R}$?

Assignment.

1. $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denoted by

$d(P,Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

where $P = (x_1, y_1)$, $Q = (x_2, y_2)$

2. $d(x,y) = \frac{1}{3}|x-y|$ $\forall x,y \in n$

3. Let X be a non void set def $d: X \times X \to \mathbb{R}$ by

$d(x,y) = \begin{cases} 0 & \text{iff } x = y \\ 1 & \text{iff } x \neq y \end{cases}$

Is d a metric on X? What is its name?

Neighbourhood

Let (X,d) be a MS let $x_0 \in X$ and $\varepsilon > 0$ be any +ve real no. $N(x_0, \varepsilon)$

$$N(x_0, \varepsilon) = \{x \in X : d(x, x_0) < \varepsilon\}$$

x_0 is called the Centre of the Neighbourhood.

ε - is called the radius of the Neighbourhood

In $(\mathbb{R}, d)$

$$N(x_0, \varepsilon) = \{x \in \mathbb{R} : d(x, x_0) < \varepsilon\}$$
$$= (x_0 - \varepsilon, x_0 + \varepsilon)$$

Example

If $x_0 = 5$ and ε 1

$$(5-1, 5+1)$$
$$= (4, 6)$$

Interior point

Let (X,d) be a MS and $A \subseteq X$. A point $P \in A$ is called an interior ~~metric~~ points of A if $\exists$ neighbourhood $N(P,\varepsilon)$ set $N(P,\varepsilon) \subseteq A$.

The set of all interior points of A is called the interior of A and is denoted by

$\mathrm{Int}(A)$ or A^{0}

A point $Q \in X$ is called an exterior point of A, if $Q \in \mathrm{Int}\, A^{c}$; here $A^{c} = X \setminus A$

Remarks

$x_0 \in X$, $\varepsilon > 0$, $N(x_0, \varepsilon)$, $x \in X : d(x, x_0) < \varepsilon$

If a point $\mathbb{R}$ is neither interior nor exterior then is called a boundary point.

Explain interior, exterior and boundary.

Open Set

Let (X,d) be a Metric Space and $A \subseteq X$ - A is said to be open if it contains all the interior points.

Closed Set

A set is said to be closed if it contains all its limit points. or if is the complete of A. (A^{c})

Assignment

Prove that the empty set is both closed and open.

Let (X,d) be a metric space and $A \subseteq X$, A is said to be open if it contain all interior point

Open

Consider empty set ϕ. There is no interior point in ϕ hence its oped.

Close

The complement ϕ is X itself. and ~~empty~~ ϕ^{c} ~~set~~ is open following that it is closed.

TOPOLOGICAL SPACE

Let X be a non-empty set; A family $\mathcal{T}$ of all open subsets of X is called a topology on X if the following conditions are satisfied.

1) ϕ and X belongs to $\mathcal{T}$

2) A arbitrary union of members of $\mathcal{T}$ belongs to $\mathcal{T}$

3) finite intersection of members of $\mathcal{T}$ belongs to $\mathcal{T}$

The ordered pair $(X, \mathcal{T})$ is called a topological Space.

Example

Let $X = \{1, 2, 3\}$ and $\mathcal{T}_s = \{\phi, \{1\}, X\}$ is $\mathcal{T}_s$ a topology on X?

Soln

i) Clearly ϕ and X belongs to $\mathcal{T}_s$
$\phi, X \in \mathcal{T}_s$

ii) $\phi \cup \{1\} = \{1\} \in \mathcal{T}_s$
$\phi \cup X = X \in \mathcal{T}_s$
$\{1\} \cup X = X \in \mathcal{T}_s$

iii) $\phi \cap \{1\} = \phi \in \mathcal{T}_s$
$\phi \cap X = \phi \in \mathcal{T}_s$
$\{1\} \cap X = \{1\} \in \mathcal{T}_s$

Since all the three properties are satisfied $\mathcal{T}_s$ is a topology on X called sierprisk topology.

Example

Let X be a non-void set and $\mathcal{T}_{in} = \{\phi, X\}$ is $\mathcal{T}_{in}$ a top of X?

Clearly
i) $\phi, X \in \mathcal{T}_{in}$.

ii) $\phi \cup X = X \in \mathcal{T}_{in}$.

iii) $\phi \cap X = \phi \in \mathcal{T}_{in}$

Since all three properties are satisfie. $\mathcal{T}_{in}$ is a

$= \mathcal{T}_{in}$ is a indiscrete topology

Assignment

Describe the discrete topological space. (8 elements)

Ex

Let $X = \{1, 2, 3\}$ & $\mathcal{T} = \{\phi, \{1\}, \{1,2\}, \{1,3\}, X\}$ is $\mathcal{T}$ a topology on X?

Soln

Clearly
i) ϕ, and $X \in \mathcal{T}$.

ii) $\phi \cup \{1\} = \{1\} \in \mathcal{T}$, $\{1\} \cup \{1,2\} = \{1,2\} \in \mathcal{T}$
$\phi \cup \{1,2\} = \{1,2\} \in \mathcal{T}$ $\{1\} \cup \{1,3\} = \{1,3\} \in \mathcal{T}$
$\phi \cup \{1,3\} = \{1,3\} \in \mathcal{T}$ $\{1,2\} \cup \{1,3\} = X \in \mathcal{T}$
$\phi \cup X = X \in \mathcal{T}$

iii) $\phi \cap \{1\} = \phi \in \mathcal{T}$
$\phi \cap \{1,2\}\ \phi \in \mathcal{T}$
$\phi \cap \{1,3\} = \phi \in \mathcal{T}$
$\phi \cap X = \phi \in \mathcal{T}$
$\{1,3\} \cap \{1,2\} = \{1\} \in \mathcal{T}$
$\{1,3\} \cap \{1,3\} = \{1,3\} \in \mathcal{T}$

Since all the three properties are satisfied τ is topology on X.

Example 2

Let $X = \{1\ 2\ 3\}$ and $\tau = \{\phi, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$

Is τ a topology on X?

not sure... $13 \cap 23 = \{3\} \notin \tau$

$12 \cap 23 = 2 \notin \tau$.

Assignment.

1 Describe the following topologies

I. A - inclussion topology
II. A - Exclusion topology.
III. Co-finite topology

2. Describe the following in a top. Space.

i) interior of a Set.
ii) Exterior of a Set.
iii) limit point.
iv) Closure of a Set.
V. Subspace topology
VI. Base.

Solution

Describe the following topologies.

I - Inclusion / Inclusion topology.
- Given a set x and a subset $A \subseteq X$, the inclusion topology on x (with respect to A) is the topology whose open set are ϕ, x, and all Subsets of A.

$\Rightarrow$ Every subset of A is open and the only other open set is X itself (if $A \neq X$)

II Exclusion topology [A-exclusion]

- Given a set X and a subset $A \subseteq X$, the exclusion topology is the topology whose open set are ϕ and all subsets of X that Contain A.

$\Rightarrow A$

IV) Subspace topology (Relative topology).

- Given a topological space (x, τ) and a subset $Y \subseteq X$ the subspace topology on Y is defined by

$\tau_y = \{U \cap Y : U \in \tau\}$.

$\Rightarrow$ A set is open in Y iff it is the intersection of Y with an open set of X.

V) Base (Basis) of a topology

- A collection B of open subset of a topological space (X, τ) is a base of τ if every open set in τ can be written as a union of element of B.

$\Rightarrow$ The base generates the topology by taking arbitrary unions of its members.

Describing the following in a topological Space:

1) Interior of a Set A (int(A))

- The interior of A is the largest open set contained in A.

⇒ A point $x \in Int(A)$ if $\exists$ an open neighbourhood U of x such that $U \subseteq A$.

2) Exterior of a Set A (Ext(A))

- The exterior of A is the interior of the compliment of A (ie $Ext(A) = Int(X \setminus A)$), where X is the whole space.

⇒ Ext(A) is the set of all points that have an open neighbourhood entirely outside A.

3) Limit point (Accumulation point) of a Set A

- A point $x \in X$ is a limit point of A if every open neighborhood of X contains atleast once point of A different from x itself.

4) Closure of a set A [CL(A) or $\bar{A}$]

- Closure of A is the Smallest Closed set containing A.

⇒ $CL(A) = A \cup \{$all limit points of A$\}$ or $CL(A) = Int(A) \cup Bd(A)$, where Bd(A) is the boundary of A.

N/B: $Bd(A) \cup Ext(A)$ (disjoint union).

Revision

Metric Spaces.

Q1. Show that the following $d: X \times X \to \mathbb{R}$ defined as

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Q2. Let $X = \{x, y, z\}$ and $\tau = \{\phi, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$

Determine whether τ is a topology on X or not.

Soln

⇒ X & $\phi \in \tau$.

i) Clearly X & $\phi \in \tau$

ii) Union.

$\phi \cup \{y\} = y \in \tau$; $\{z\} \cup \{x,z\}$ $xz \in \tau$

$\{z\} \cup \{y,z\}$ $yz \in \tau$

$\phi \cup \{z\} = z \in \tau$; $\{z\} \cup X$ · $X \in \tau$

$\phi \cup \{x,z\} = xz \in \tau$

$\phi \cup \{y,z\}$ $yz \in \tau$; $\{x,z\} \cup \{y,z\}$ $X \in \tau$

$\phi \cup X$ $X \in \tau$; $\{x,z\} \cup X$ $X \in \tau$

$\{y,z\} \cup X$ $X \in \tau$

$\{y\} \cup \{z\}$ $\{y,z\} \in \tau$

$\{y\} \cup \{x,z\}$ $\{x,y,z\} = X \in \tau$

$\{y\} \cup \{y,z\}$ $\{y,z\} \in \tau$

$\{y\} \cup X = X \in \tau$

iii) Intersection

$\phi \cap \{y\} = \phi \in \tau$; $\{z\} \cap \{x,z\} = z \in \tau$

$\phi \cap \{z\} = \phi \in \tau$; $\{z\} \cap \{y,z\} = y \in \tau$

$\phi \cap \{x,z\}$ $\phi \in \tau$; $\{z\} \cap X = z \in \tau$

$\phi \cap \{y,z\}$ $\phi \in \tau$

$\phi \cap X = \phi \in \tau$

$\{x,z\} \cap \{y,z\} = z \in \tau$

$\{y\} \cap \{z\} = \phi \in \tau$; $\{x,z\} \cap X = x,z \in \tau$

$\{y\} \cap \{x,z\}$ $\phi \in \tau$; $\{y,z\} \cap X$ · $yz \in \tau$

$\{y\} \cap \{y,z\}$ $y \in \tau$

$\{y\} \cap X = y \in \tau$

Since all these properties are satisfied τ is a topology on X

Q3 2°

Let $X = \{1\,2\,3\}$ and $\tau = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, X\}$

If $A = \{1, 3\}$ find.

$\{\phi\ 1, 3\}\ \{1\ 3\}$

i) Int (A) — largest open set contained in A.

ii) EXT (A) largest open set contain in A^c

iii) Boundary of A.

i) $\phi, \{1\}\ \{1, 3\}\ \{3\}$

$Int(A) = \{1\}$

$A = (\phi, 1\,3, 1, 3)$

$(1\,2\,3)/(1\,3)$.

ii) $EX(A)$

X/A

$EX(A) = Int(X/A)$

$X/A = \{1\,2\,3\}/\{1\,3\} = \{2\}$

$EX(A) - int\{2\}$

$\{\phi\}, \{2\}$

$Ext(A) = \{2\}$

iii) Boundary set in A that is not contained in

Boundary $= \{3\}$

iv) Closure of $A = \bar{A} = In(A) \cup Bd(A)$
$= \{1\} \cup 3 = \{1\,3\}$

Q4

Given that $Y = \{x, y, z\}$, $\tau_1 = \{\phi, \{z\}, \{z\}, \{x, z\}, \{x, y\}, Y\}$

and $\tau_2 = \{\phi, \{x\}, \{x, y\}, \{x, z\}\ Y\}$

Determine whether $\tau_1 \cup \tau_2$ is a topology on Y or not.

$\Rightarrow \tau_1 \cup \tau_2$

$\tau = \{\phi, \{x\}, \{z\}, \{x, y\}, \{x, z\}\ Y\}$

Clearly

i) ϕ and $Y \in \tau$

ii) Union

$\phi \cup \{x\} = \{x\} \in \tau$ $\quad \{x\} \cup \{z\} = \{x, z\} \in \tau$

$\phi \cup \{z\} = \{z\} \in \tau$ $\quad \{x\} \cup \{x, y\} = \{x, y\} \in \tau$

$\phi \cup \{x, y\} = \{x, y\} \in \tau$ $\quad \{x\} \cup \{x, z\} = \{x, z\} \in \tau$

$\phi \cup \{x, z\} = \{x, z\} \in \tau$ $\quad \{x\} \cup Y = Y \in \tau$

$\phi \cup Y = Y \in \tau$

$\{z\} \cup \{x, y\} = Y \in \tau$ $\quad \{x, y\} \cup \{x, z\} = Y \in \tau$

$\{z\} \cup \{x, z\} = \{x, z\} \in \tau$ $\quad \{x, y\} \cup Y = Y \in \tau$

$\{z\} \cup Y = Y \in \tau$ $\quad \{x, z\} \cup Y = Y \in \tau$

iii) $\phi \cap \{x\} = \phi \in \tau$ $\quad \{x\} \cap \{z\} = \phi \in \tau$

$\quad$ " $\{z\}$ = " $\quad \{x\} \cap \{x, y\} = \{x\} \in \tau$

$\quad$ " $\{x, y\}$ = " $\quad \{x\} \cap \{x, z\} = \{x\} \in \tau$

$\quad \{x, z\}$ = " $\quad \{x\} \cap Y = \{x\} \in \tau$

$\quad Y$ = "

$\{z\} \cap \{x, y\} = \phi \in \tau$

$\{z\} \cap \{x, z\}\ \{z\} \in \tau$

$\{z\} \cap Y\ \{z\} \in \tau$

$\{x, y\} \cap \{x, z\} = \{x\} \in \tau$

$\{x, y\} \cap Y = \{x, y\} \in \tau$

$\{x, z\} \cap Y = \{x, z\} \in \tau$

Since all the properties are satisfied $\tau_1 \cup \tau_2$ is a topology on Y.

Q5

Let $\{X, d\}$ be a Metric space. Show that the empty space is both Open and Closed.

Open set:

let (X, d) be a metric space an $A \subseteq X$. A is said to be open if it contains all interior points.

Open

For ϕ there is no $x \in \phi$, so the condition for every $x \in \phi$, is vacuously true.

Closed.

The complement of ϕ in X is X itself. Since X is a metric space, X is open. Therefore, the complement of ϕ is open $\Rightarrow \phi$ is closed.

Q6.
1) Define a topological space as used in topology.

1. State and prove Demorgans theorem

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$.

1. $x \notin A \cup B$
$x \notin A$ and $x \notin B$
$\Rightarrow x \in A^c$ and $x \notin B^c$
$\Rightarrow x \in A^c \cap B^c$

Reverse
$x \in A^c \cap$

2. $(A \cap B)^c = A^c \cup B^c$

$x \notin (A \cap B)$
$x \notin A$ or $x \notin B$
$x \in A^c$ or $x \in B^c$,
$x \in A^c \cup B^c$

Q7. Concider $B = \{a, b, c, d\}$. Is $\beta = \{\{a, d\}, \{d\}, \{b, c, d\}\}$ a base for any topology on B? explain.

✓ Power sets $= 2^n$
✓ Cadinallity = no of ements in a set.

Revision

1. $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denoted by
$d(p, q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ CL1
where $p = (x_1, y_1)$ $q = (x_2, y_2)$

$d(x_1 y_1, x_2 y_2) = d(p, q)$
$d(p, q)^2 = \sqrt{(p-q)^2 + (p-q)^2}^2$
$d(p^2 q^2) = \sqrt{2(p-q)^2}$
$d(p^2, q^2) = 2(p-q)^2$

i) Clearly
$d(p^2, q^2) = 2(p-$

March 30th, 2022

SEPARATION AXIOMS

T_0 Axiom

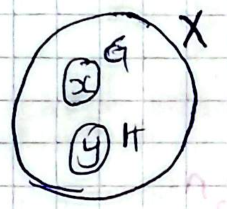
$x, y \in X$, $x \neq y$ $\exists$ an open set G. Such that $x \in G$ and $y \notin G$.

- A topological space satisfied by T_0 axioms is called a T_0 space.

T_1 axiom

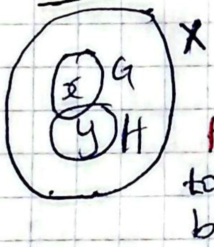
For every pair of distinct points $(x, y) \in X$ $\exists$ opensets G and H such that $x \in G$, $y \notin G$, $y \in H$ and $x \notin H$.

- A topological Space Satisfied by T_1 axiom is called a T_1 Space.



T_2 axiom

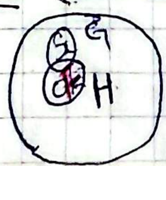
For every pair of distinct points $x, y \in G$ $\exists$ open sets G and H $\in X$, Such that $x \in G$ and $y \in H$ and $G \cap H = \emptyset$



Haustraff Space is a topological space satisfied by T_2 axiom.

T_3 axiom (regular T_1)

For every pair of distinct pairs $x, y \in X$ and every closed sets $F \in X$ and $x \notin X$ $\exists$ an open set G and $H \in X$ Such that $x \in G$. $F \in H$



Question

Let $X = \{a, b, c\}$ $\mathcal{I} = \{\emptyset, \{a\}, \{b, c\}, X\}$

i) $(X, \mathcal{I})$ is a regular space

ii) $(X, \mathcal{I})$ is a T_2 space.

Soln

T_2-space, For every $x, y \in X$ $x \neq y$

ii) $\exists$ open sets G & H $x \in G$ and $y \in H$

Let $x = a$, $y = b$, $a, b \in X$ $a \neq b$

$G = \{a\}$, $x \in \{a\}$, $y \in \{b, c\}$

$\{a\} \cap \{b, c\} = \emptyset$

$\Rightarrow (X, \mathcal{I})$ is not T_2 space

i) solution

closed sets.

$\emptyset^c = X$

$\{a\}^c = \{b, c\}$

$\{\emptyset, \{a\}, \{b, c\}, X\}$

$\{b, c\}^c = \{a\}$

$X^c = \emptyset$

$x \in G$, $F \subset H$, $G \cap H = \emptyset$

Question 2

Let $X = \{1, 2\}$ $\mathcal{I}_s = \{\emptyset, \{1\}, X\}$ is T_0-space.

Soln

T_0-space; $x, y \in X$ & $x \neq y$ $\exists$ open set G. s.t $x \in G$, $y \notin G$

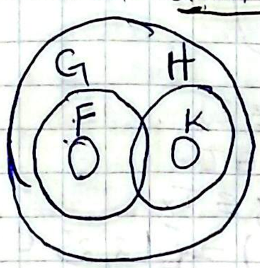
$x = 1$, $y = 2$, $1, 2 \in X$, $1 \neq 2$.

$G = \{1\}$, $x \in \{1\}$, $y \notin \{1\}$

is $(X, \mathcal{I})$ a T_0 Space

T_4 axiom (Normal T_1)

For every pair of disjoint closed sets F and K $\in X$ $\exists$ an open sets G and H such that $F \subset G$, $K \subset H$ and $G \cap H = \phi$



Regular Space

- A topological space $(x \in \tau)$ is said to be regular if it's satisfying a T_3 axiom.

T_3 space

- A regular T_1 space is called a T_3 space.

Tychonoff space

- A completely regular space is called tychonoff space

N/B: Every tychonoff is a regular space but the converse is not true

Normal space

- A topological space $x \in \tau$ is said to be normal if it satisfies a T_4 axiom and it is a T_1 space.

T_4 Space

- Is a topological space which is normal

Regular space $\leftarrow$ T_3 axiom $+ T_1$ $\rightarrow$ T_3 space

Normal space $\leftarrow$ T_4 axiom $+ T_1$ $\rightarrow$ T_4 space

Example

Let $X = \{1, 2\}$ and $\tau_s = \{\phi, \{1\}, X\}$ is τ_s a T_0 space?

Soln

Yes indeed, in τ_s $\exists$ $\{1\}$ containing $1 \in X$ and we have $2 \in X$ such that $2 \notin \{1\}$

$x = 1$, $y = 2$, $1, 2 \in X$, $1 \neq 2$, $G = \{1\}$, $x \in \{1\}$, $y \notin \{1\}$

*So the equation is containing 1 in $\{1\}$

EXE

Let x be any set and D be a topological space X. Is (x, D) a ~~top~~ T_0 space.

Remark

- Every T_1 space is a T_0 space but the ~~the~~ converse is not true.

Example

Concider D (sierpnski space)

$$\left[\begin{array}{l} X = \{1, 2\} \\ T_s = \{\phi, \{1\}, X\} \end{array}\right.$$

$1 \in \{1\}$, $2 \notin \{1\}$

We can not find if such $2 \in H$, $1 \notin H$ such that hence the sierpnski space is T_0 and not T_1

$$\tau_0 = \{\phi, \{1\}, \{2\}, X\}$$

$1 \in \{1\}$, $2 \in \{2\}$

Prove that a topological space $(X, \mathcal{T})$ is a T_1 space iff every single.

Remark

$T_2 \Rightarrow T_1 \Rightarrow T_0$

Conversely is not true.

Every T_2 is T_1 but T_1's not T_2

Assignment:

Prove the following:

1) Every Hausdorff is a T_1 Space.
2) Every metric space is a T_2 Space
3) Every Subspace of a T_2 space is " Hausdorff
4) Every convergent Sequence in a T_2 space converges to a unique limit.

CONTINUITY IN TOPOLOGICAL SPACES

Taking $f: X \to Y$ f is Continuous if $\exists f^{-1} \ni u \in Y$ such that $f^{-1}(u)$ is open in X. $p \in X$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. A function $f: X \to Y$ if $\forall u \in \mathcal{T}_Y, f^{-1}(u) \in \mathcal{T}_X$

When u is open in Y, $f^{-1}(u)$ is open on X.

Example

Concider $X = \{1, 2, 3, 4\}$ and $Y = \{5, 6, 7, 8\}$

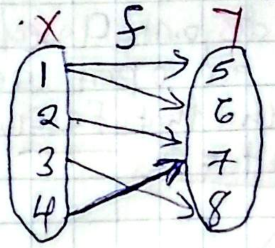
$\mathcal{T}_X = \{\phi, \{1\}, \{1, 2\}, \{1, 2, 3\}, X\}$

$\mathcal{T}_Y = \{\phi, \{5\}, \{6\}, \{5, 6\}, \{6, 7, 8\}, Y\}$

Let $f: X \to Y$ defined as $f(1) = (6)$, $f(2) = 7$, $f(3) = 8$, $f(4) = 7$

Is f a continuous Function?

Soln



$f^{-1}(5) = \phi \in \mathcal{T}_X$

$f^{-1}(6) = \{1\} \in \mathcal{T}_X$

$f^{-1}(5, 6) = \{1\} \in \mathcal{T}_X$

$f^{-1}(5, 6, 7, 8) = \{1, 2, 3\} \in \mathcal{T}_X$

$\therefore f$ is continuous function.

Example 2

Let $X, Y, \mathcal{T}_X, \mathcal{T}_Y$ be as in example 1 above.

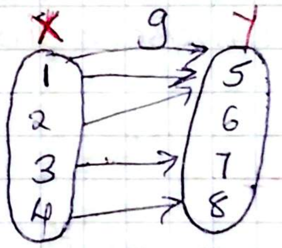
Let $g: X \to Y$ be defined as $g(1) = 5$, $g(2) = 5$, $g(3) = 7$, $g(4) = 8$ is a Continuous Function?

Soln

$X = \{1, 2, 3, 4\}$, $Y = \{5, 6, 7, 8\}$

$\mathcal{T}_X = \{\phi, \{1\}, \{1, 2\}, \{1, 2, 3\}, X\}$

$\mathcal{T}_Y = \{\phi, \{5\}, \{6\}, \{5, 6\}, \{6, 7, 8\}, Y\}$



$g^{-1}(\phi) = \phi \in \mathcal{T}_X$

$g^{-1}(5) = \{1, 2\} \in \mathcal{T}_X$

$g^{-1}(6) = \phi \in \mathcal{T}_X$

$g^{-1}(5, 6) = \{1, 2\} \in \mathcal{T}_X$

$g^{-1}(6, 7, 8) = \{3, 4\} \notin \mathcal{T}_X$

$\therefore g$ is not Continuous

Since $g^{-1}(6, 7, 8)$ does not belong to $\mathcal{T}_X$ then g is not Continuous

HOMEOMORPHISMS

Def Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be topological spaces.

An open and continuous function $f: X \to Y$ is called a homeomorphism i.e a function ~~with~~ which is open, continuous 1-1, onto and inverse. ~~If a~~ $\exists$ a homeomorphism on X onto Y, then $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ is said to be a homomorphism.

Example 2

Let $A = [0,5]$, $I = [0,1]$ then

$f: I \to A$ defined by

$f(x) = 5 \quad \forall x \in I$ is a holmophism

I is homormophism on A.

Assignment

1) Define an i) isometry
ii) First countable topological space
iii) 2^{nd} countable " "

2) Describle topological subspaces and show that a property being T_2 is hereditary.

Completely Regular Space - Tychonoff Space

A topological space $(X, \mathcal{I})$ is said to be completely regular if for any closed subset A in X and $x \in X$ s.t $x \notin A$ $\exists$ a continous function $f: X \to [0,1]$ s.t $f(x) = 0$ and $f(A) = [1]$.

~~Revision~~

Assignments Solutions

A topological space $(X, \mathcal{I})$ is a T_1-space iff every singleton of X is closed

Proof

i) Suppose $(X, \mathcal{I})$ is a T_1 space

We have to show $[p]^c$ is open $p \in X$

Let $x \in [p]^c$ then $x \neq p$

Since $(X, \mathcal{I})$ is a T_1 space $\exists$ an open set G_x s.t $x \in G_x$ but $p \notin G_x$

Hence $x \in G_x$ ~~$\subset [p]^c$~~ but $p \notin G_x$

Hence $x \in G_x \subset [p]^c$ hence

$[p] = \cup [G_x : x \in [p]^c]$

We know that union of open sets is open.

Alternative

1) Let every singleton $\{x\}$ of X be close

We have to prove that $(X, \mathcal{I})$ is a T_1 space

$\{x\}^c$ is open

Let x, y be 2 arbitrary distinct points of X s.t $\{x\}$ and $\{y\}$ are closed

So $\{x\}^c$ and $\{y\}^c$ are open

$x \in \{y\}^c$ ~~and~~ but $y \notin \{y\}^c$

$y \in \{x\}^c$ but $x \notin \{x\}^c$

Hence by def $(X, \mathcal{I})$ is a T_1-Space

Assignments

Describe the discrete topological space. Is a topological space in which every subset is open.

Example.

If x be a set the descrete topology on x is τ given by P_x, τ_{px}. That is all subsets of x including both empty and x are declaired open.

If $x = abc$ the decrete topology $\mathcal{P}_x$ is $= \{\phi, \{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{bc\}, X\}$

Let $x = (1\ 2\ 3)$ $\tau = \{\phi, \{1\}, \{2\}, \{1,3\}, \{2,3\}, X\}$

Determine whether τ is a topology on x or not.

i) Clearly.

$\phi, X \in \tau$.

ii) Arbitrary union.

$\phi \cup \{1\} = 1 \in \tau$

$\phi \cup \{2\} = 2 \in \tau$

$\phi \cup \{1,3\} = \{1,3\} \in \tau$

$\phi \cup \{2,3\} = \{2,3\} \in \tau$.

$\phi \cup X = X \in \tau$

$\{1\} \cup \{2\} = \{1,2\} \notin \tau$

$\{1\} \cup \{1,3\} = 13 \in \tau$

$1 \cup \{2,3\} = 123 = X \in \tau$

$1 \cup X = 123 = X \in \tau$

$\{2\} \cup \{1,3\} = \cancel{123} = X \in \tau$

$2 \cup X = X \in \tau$

$2 \cup 23 = \{2,3\} \in \tau$

$\{1,3\} \cup \{2,3\} = X \in \tau$

$13 \cup X = X \in \tau$

iii) Finite intersection

$\phi \cap \{1\} = \phi \in \tau$	$\{1\} \cap \{2\} = \phi \in \tau$
$\phi \cap \{2\} = \phi \in \tau$	$\{1\} \cap \{1,3\} = \{1\} \in \tau$
$\phi \cap \{1,3\} = \phi \in \tau$	$\{1\} \cap \{2,3\} = \{\phi\} \in \tau$
$\phi \cap \{2,3\} = \phi \in \tau$	$\{2\} \cap \{1,3\} = \phi \in \tau$
$\phi \cap X = \phi \in \tau$	$\{2\} \cap \{2,3\} = 2 \in \tau$
	$\{2\} \cap X = \{2\} \in \tau$.

$\{1,3\} \cap \{2,3\} = \{3\} \notin \tau$

$\{1,3\} \cap X = 13 \in \tau$

$\{2,3\} \cap X = \{2,3\} \in \tau$

---

1. $X = \{1\ 2\ 3\ 4\ 5\}$ $\tau = \{\phi, \{1\}, \{3\}, \{1,3\}, \{3,4\}, \{1,3,4\}, X\}$.

If $A = \{1\ 2\ 4\}$

Find the subspace topology τ_A on A.

$$\tau_A = \{U \cap A : U \in \tau\}$$

Subspace topology

$A \subseteq X$

$(U \cap A)$ $U \in \tau$

perform intersection

$\phi \cap A = \phi$

$1 \cap A = 1$

$3 \cap A = \phi$

$1,3 \cap A = 1$

$3,4 \cap A = 4$

$1,3,4 \cap A = 1,4$

$X \cap A = 1,2,4$

removing repetition.

$\phi, \{1\}, \{4\}, \{1,4\}, \{1\ 2\ 4\}$

Discrete Metric Space

let X be a non empty set and $d : X \times X \to \mathbb{R}_0$ defined as:

$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & x \neq y \end{cases}$. Then show that d is metric on X.

1) $d(x,y) = 0$ if $x=y$ and

$d(x,y) = 1$ if $x \neq y$

$\Rightarrow d(x,y) \geq 0$.

ii) $d(x,y) = 0$ if $x=y$ By definition

iii) $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & x \neq y \end{cases}$

$$= \begin{cases} 0 & \text{if } y = x \\ 1 & y \neq x \end{cases}$$

$= d(y,x)$

(iv) $d(x,z) \leq d(x,y) + d(y,z)$

CASE 1 $x \neq y \neq z$

$d(x,y) = 1,\ d(y,z) = 1,\ d(z,x) = 1$

$1 < 1 + 1$

$\Rightarrow d(x,z) < d(x,y) + d(y,z)$ — (i)

CASE II $x \neq y = z$

$d(x,y) = 1\ d(y,z) = 0,\ d(z,x) = 1$

$1 = 1 + 0$

$d(x,z) = d(x,y) + d(y,z)$

case III

$x = y = z$

$d(x,y) = d(x,y) + d(y,z)$

$d(x,y) = 0,\ d(y,z) = 0,\ d(z,x) = 0$

$0 = 0 + 0$

from ① ② and ③ we conclude

$d(x,z) \leq d(x,y) + d(y,z)$.

Example 2

For set $R^2 = \{(x_1,x_2), (y_1,y_2)\}$

$d(x_1,x_2)(y_1,y_2) = |x_1 - y_1| + |x_2 - y_2|$

Solution

As $|x_1 - y_1| + |x_2 - y_2| \geq 0$

i) $d(x_1,x_2)(y_1,y_2) \geq 0$

ii) $d(x_1,x_2)(y_1,y_2) = |x_1 - y_1| + |x_2 - y_2|$

$= |y_1 - x_1| + |y_2 - x_2|$

$= d(y_1,y_2)(x_1,x_2)$

iii) $d(x_1,x_2)(y_1,y_2)$

$\Leftrightarrow |x_1 - y_1| = 0,\ |x_2 - y_2| = 0$

$\Leftrightarrow x_1 - y_1 = 0,\ x_2 - y_2 = 0$

$\Leftrightarrow x_1 = y_1,\ x_2 = y_2$

$\Leftrightarrow (x_1,x_2) = (y_1,y_2)$

iv) $d(x_1,x_2),(z_1,z_2) = |x_1 - z_1| + |x_2 - z_2|$

$= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2|$

$\leq (|x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|)$

$d(x_1,x_2),(z_1,z_2)) \leq d((x_1,x_2),(y_1,y_2)) + d((y_1,y_2)(z_1,z_2))$.

(X,d) is a metric space on

Openset in topological space

let (X,τ) be a topological space and $A \subseteq X$ then A is said to be open set if $A \in \tau$

Example

if $x = \{1,2,3\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$ is a topology on X then $\{1,2\}$ is open set in (X,τ)

But $\{1,3\}, \{2,3\}, \{3\}$ are not open set in τ.

Theorem

Let (X,τ) be a topological space. The

i) ϕ and X are open sets.

ii) union of any number of opens sets are open.

iii) intersection of finite number of open sets are open

proof

i) As $\phi \in \tau \Rightarrow \phi$ is open

As $X \in \tau \Rightarrow X$ is open.

ii) Let $\{A_\alpha : \alpha \in I\}$ be collection of any number of open sets

To prove $\bigcup_{\alpha \in I} A_\alpha$ is open.

As A_α is collection of open sets

So A_α is open.

$\Rightarrow A_\alpha \in \mathcal{I}$

$\Rightarrow \cup A_\alpha \in \mathcal{I}$ [$\mathcal{I}$ is topology on X]

$\Rightarrow \cup A_\alpha$ is open.

(ii) Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of open sets.

To prove

$\bigcap_{i=1}^{n} A_i$ is open

As A_i is open $\Rightarrow A_i \in \mathcal{I}$

$\Rightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{I}$ ($\mathcal{I}$ is topology on X)

$\Rightarrow \bigcap_{i=1}^{n} A_i$ is open. $\boxed{}$

Hence proved

~~Closure of A set~~
~~Let $(X, \mathcal{I})$ be a topological space and $A \subseteq X$, then, Closure of A, denoted by $\bar{A}$ is the intersection of all closed supersets of A.~~

Closed Set

Let $(X, \mathcal{I})$ be a topological space and $A \subseteq X$, the A is said to be closed if and only if A^c is open.

A' in place of A^c is used in open set.

Closure of a Set

Let $(X, \mathcal{I})$ be a topological space and $A \subseteq X$, Then, Closure of A, denoted by $\bar{A}$ is the intersection of all the closed supersets of A.

Example

Let $X = \{a, b, c, d\}$, $\mathcal{I} = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\}$

$A = \{b, c\}$ then Find $\bar{A}$

Solution

Closed sets are: $X, \phi, \{b, c, d\}, \{c, d\}, \{d\}$.

closed supersets of A are:

$X, \{b, c, d\}$

$\bar{A} = X \cap \{b, c, d\}$

$= \{b, c, d\}$